

Lecture 21 (April 18th, 2016)

Summary.

$$\dot{x} = f(x, u) \quad f(0, 0) = h(0, 0) = 0$$

$$y = h(x, u) \quad f: \text{locally Lip}, \quad g: \text{cont.}$$

V: storage function, positive semi definite

$$\rightarrow \text{Passive: } u^T y \geq \dot{V}$$

$$\rightarrow \text{strictly passive: } u^T y \geq \dot{V} + \psi(x), \quad \psi: \text{positive definite}$$

$$\rightarrow \text{output strictly passive} \quad u^T y \geq \dot{V} + y^T \varphi(y), \quad \varphi(y) > 0 \quad (\text{excess of passivity})$$

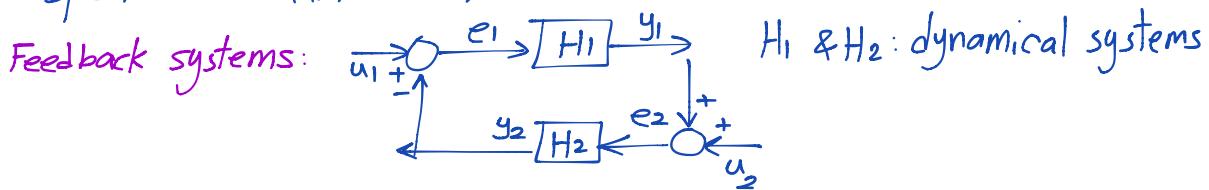
$$\varphi(y) < 0 \quad (\text{shortage of P.})$$

$$\text{special case } \varphi(y) = \alpha y, \quad \alpha > 0$$

$$\rightarrow \text{input strictly passive} \quad u^T y \geq \dot{V} + u^T \varphi(u), \quad \varphi(u) > 0 \quad (\text{excess of passivity})$$

$$\varphi(u) < 0 \quad (\text{shortage of P.})$$

$$\text{special case } \varphi(u) = \alpha u, \quad \alpha > 0$$



Thm 6.1. H_1 & H_2 : Passive $\rightarrow H$: passive $\rightarrow \theta$: stable

Thm 6.2 $e_i^T y_i \geq \dot{V}_i + \varepsilon_i e_i^T e_i + \varsigma_i y_i^T y_i, \quad \varepsilon_1 + \varepsilon_2 > 0, \quad \varsigma_1 + \varsigma_2 > 0$
 $\rightarrow \theta$: finite gain L_2 stable

Thm 6.3. θ : a.s if

H_1 & H_2 : strictly passive

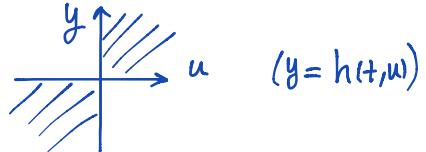
H_1 & H_2 : output strictly passive & zero state observable

H_1 : s.p & H_2 : o.s.p & z.s.o

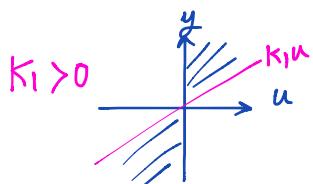
Def. $h(t, u)$ is called Memoryless if its value at any instant of time is determined uniquely by its input at that instant; it doesn't depend on the history of the input. e.g. $h(t) = u(t)^2$.

Def: A memoryless function $h: [0, \infty) \times \mathbb{R}^p \rightarrow \mathbb{R}^p$ is said to belong to the sector ($y = h(t, u)$)

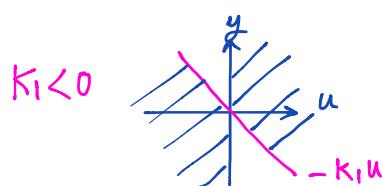
1) $[0, \infty]$ if $u^T h(t, u) \geq 0$ (Passive)



2) $[k_1, \infty]$ if $u^T h(t, u) \geq k_1 u^T u$ ($k_1 > 0$: input strictly passive)



excess of passivity



shortage of passivity

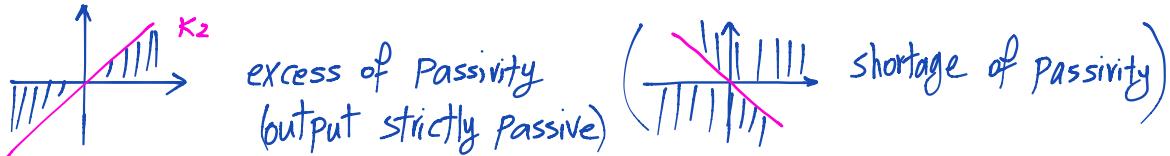
removal of excess or shortage:

$$\begin{array}{c} u \xrightarrow{\quad} y \\ h: (u, y) \\ h \in [k_1, \infty] \end{array} \quad \begin{array}{c} \tilde{u} \xrightarrow{+u} \xrightarrow{k_1} y \xrightarrow{-} \tilde{y} \\ \tilde{h}: (\tilde{u}, \tilde{y}) \\ \tilde{u}^T \tilde{y} = \tilde{u}^T (y - k_1 u) \geq 0 \\ \rightarrow \tilde{h} \in [0, \infty] \end{array}$$

$$\tilde{y} = y - k_1 u$$

$$\tilde{u} = u$$

3) $[0, k_2]$ with $k_2 = k_2^T > 0$ if $u^T h(t, u) \geq k_2 h^T(t, u) h(t, u)$



excess of Passivity
(output strictly passive)

(shortage of passivity)

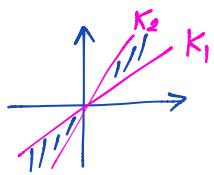
removal of excess or shortage:

$$\begin{array}{c} u \xrightarrow{\quad} y \\ h: (u, y) \\ h \in [0, k_2] \end{array} \quad \begin{array}{c} \tilde{u} \xrightarrow{+u} \xrightarrow{k_2} y \xrightarrow{-} \tilde{y} \\ \tilde{h}: (\tilde{u}, \tilde{y}) \\ \tilde{u}^T \tilde{y} = (u - k_2 h)^T h \geq 0 \\ \rightarrow \tilde{h} \in [0, \infty] \end{array}$$

$$\tilde{y} = y$$

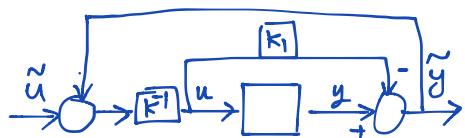
$$u = \tilde{u} + k_2 h \rightarrow \tilde{u} = u - k_2 h$$

4) $[K_1, K_2]$ with $K = K_2 - K_1 = K^T > 0$ if $(h - K_1 u)^T (h - K_2 u) \leq 0$



$$h: (u, y) \rightarrow \boxed{\quad} \rightarrow y$$

$$h \in [K_1, K_2]$$



$$\tilde{h}: (\tilde{u}, \tilde{y}) \rightarrow \tilde{y}$$

$$\begin{aligned}\tilde{y} &= y - K_1 u \\ K^T(\tilde{u} + \tilde{y}) &= u\end{aligned}$$

$$\rightarrow \tilde{u} = Ku + K_1 u - y = K_2 u - y$$

$$\tilde{u}^T \tilde{y} = (K_2 u - y)^T (y - K_1 u)$$

$$= \underbrace{-(y - K_2 u)^T (y - K_1 u)}_{\leq 0} \geq 0$$

H_1 : dynamical system x_1, e_1, y_1

H_2 : memoryless function e_2, y_2

Thm 6.4. H_1 : strictly passive

$\Rightarrow \emptyset$: a.s

H_2 : passive

\emptyset : g.a.s if V_1 : radially unbdd

Proof. V_1 : candidate for Lyapunov function

$\rightarrow V_1 > 0$ (as shown in Lemma 6.7)

$\rightarrow \dot{V}_1 \leq e_1^T y_1 - \psi(x_1) = -e_2^T y_2 - \psi(x_1) \leq -\psi(x_1)$

Thm 4.9 \emptyset : a.s (g.a.s) $\left(\begin{array}{l} 0 \leq y_1 - y_2 = e_1 \\ 0 \leq y_2 + y_1 = e_2 \end{array} \right)$

Thm 6.5. H_1 : $e_1^T y_1 \geq V_1 + y_1^T P_1(y_1)$ $V_1 > 0$ H_2 : $e_2^T y_2 \geq e_2^T P_2(e_2)$

zero-state observable

If $V(P_1(v) + P_2(v)) > 0 \forall v \neq 0 \Rightarrow \emptyset$: a.s. & g.a.s. if V_1 : rad. unbdd.

Note that H_1 can have shortage of passivity ($f(y) < 0$) but φ_2 must be positive & large enough s.t. $V(\varphi_1(v) + \varphi_2(v)) > 0$.

Proof. Use V_1 as a Lyapunov function.

$V_1 > 0$ (assumption)

$$\begin{aligned}\dot{V}_1 &\leq e_1^T y_1 - y_1^T \varphi_1(y_1) = -e_2^T y_2 - y_1^T \varphi_1(y_1) \leq -e_2^T \varphi_2(e_2) - y_1^T \varphi_1(y_1) \\ &< e_2^T \varphi_1(e_2) - y_1^T \varphi_1(y_1) = 0 \text{ because } e_2 = y_1 \text{ when } u=0.\end{aligned}$$

Loop transformation

Idea: Suppose one component doesn't satisfy passivity condition.

Rewrite interconnection in equivalent way to get desired properties.

Example 6.14.

$$H_1: \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -h(x_1) + bx_2 + e_1 \\ y_1 = x_2 \end{cases} \quad H_2: y_2 = \zeta(e_2)$$

where $\zeta \in [\alpha, \beta]$, $h \in [\alpha_1, \infty]$, $b > 0$, $\alpha_1 > 0$, and $k = \beta - \alpha > 0$.

(note: $\zeta(0)=0$) (note $h(0)=0$)

We can apply both Thm 6.4 & Thm 6.5 to show O is g.a.s. for H .

Apply Thm 4.5. observe that H_1 has shortage of passivity:

Let $V_1(x) = \int_0^{x_1} h(s) ds + \frac{1}{2} x_2^2$ be the storage function of H_1 . $V > 0$ &

$$\dot{V} = h(x_1) \cdot x_2 + x_2 (-h(x_1) + bx_2 + e_1) = x_2 e_1 + b x_2^2.$$

Therefore, $\dot{V} - b y_1^2 \leq e_1 y_1$. Also when $e_1=0$, $y_1=0 \Leftrightarrow x_2=0 \Leftrightarrow h(x_1)=0 \Leftrightarrow x_1=0$ $\Rightarrow H_1$: zero-state observable.

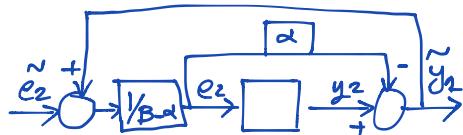
$$H_2 \in [\alpha, \beta] \rightarrow H_2 \in [\alpha, \infty] \rightarrow \underline{e_2 y_2 \geq \alpha e_2^2}$$

observe that if $\alpha > b$, then $V(-b v + \alpha v) > 0$ and by Thm 6.5. O : a.s.
 V : radially unbounded $\rightarrow O$: g.a.s.

Apply Thm 4.5. We saw that H_1 has shortage of passivity. To apply Thm 4.5, we need to remove this shortage.

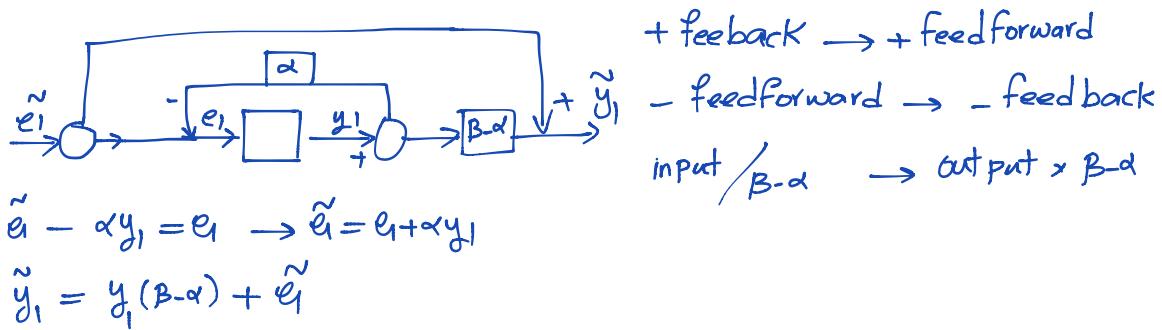
Let's make an appropriate loop transformation to reduce the passivity of H_2 as much as possible ($\alpha \in [0, \infty]$) and see what happens to H_1 .

Consider the following loop connection for H_2



We saw that this makes $\alpha \in [0, \infty] \Rightarrow \tilde{H}_2$ is (still) passive.

and consider the following for H_1 :



$$\begin{aligned} \dot{\tilde{H}}_1: \quad & \dot{x}_1 = x_2 \\ & \dot{x}_2 = -h(x_1) + \underbrace{bx_2 - \alpha x_2}_{(B-\alpha)x_2} + \tilde{e}_1 \\ & \tilde{y}_1 = x_2(B-\alpha) + \tilde{e}_1 \end{aligned}$$

Let $\tilde{V}(x) = (B-\alpha) \int_0^{x_1} h(\tilde{x}) d\tilde{x} + x^T P x$ be the storage function of \tilde{H}_1 , where

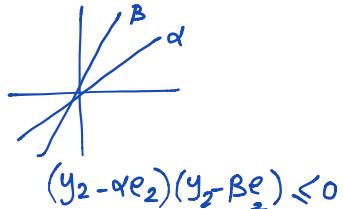
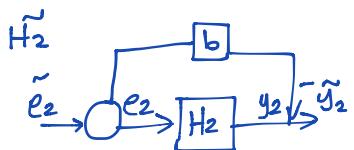
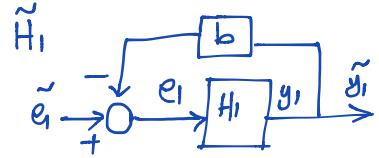
$$P = \begin{pmatrix} \alpha P_{12} & P_{12} \\ P_{12} & \frac{B-\alpha}{2} \end{pmatrix} \quad \& \quad 0 < P_{12} < \min \left\{ 2\alpha_1, \alpha \frac{B-\alpha}{2} \right\}$$

$\downarrow h \in [\alpha_1, \infty]$

and show that \tilde{H}_1 is strictly passive (exe).

$$\left. \begin{array}{l} \tilde{H}_1: \text{strictly passive} \& \tilde{V}: \text{radially unbounded} \\ \tilde{H}_2: \text{Passive} \end{array} \right\} \xrightarrow{\text{Thm 6.4}} \Omega: g.a.s.$$

① Consider the following loop transformation for both \tilde{H}_1 & \tilde{H}_2 :



$$\tilde{e}_1 - b\tilde{y}_1 = e_1$$

$$\rightarrow \tilde{e}_1 = e_1 + b\tilde{y}_1$$

$$\tilde{y}_1 = y_1$$

$$\Rightarrow \tilde{e}_1 \tilde{y}_1 = (e_1 y_1 + b\tilde{y}_1) \tilde{y}_1 \geq 0$$

$\Rightarrow \tilde{H}_1$: Passive

$$\tilde{e}_2 = e_2$$

$$\tilde{y}_2 = y_2 - b e_2$$

$$(\tilde{y}_2 - x\tilde{e}_2)(\tilde{y}_2 - y\tilde{e}_2) = \\ (y_2 - (b+x)e_2)(y_2 - (b+y)e_2)$$

If $\alpha - b \geq 0$ ($\Rightarrow \beta - b \geq 0$)

let $x = \alpha - b$, $y = \beta - b$

$$(\tilde{y}_2 - x\tilde{e}_2)(\tilde{y}_2 - y\tilde{e}_2) = (y_2 - \alpha e_2)(y_2 - \beta e_2) \leq 0$$

$$\Rightarrow \tilde{e}_2 \in [\alpha - b, \beta - b] \Rightarrow \tilde{H}_2$$
: Passive

\tilde{H}_1 & \tilde{H}_2 : Passive \Rightarrow

The interconnected system is also passive \Rightarrow The origin is stable.

Now consider a negative feedback connection with a coef greater than b , say $b + \bar{b}$, $\bar{b} = \alpha - b > 0$. Then $\tilde{e}_1 \tilde{y}_1 \geq \dot{v} + \bar{b} \tilde{y}_1^2$. Therefore \tilde{H}_1 becomes output strictly passive:

$$\begin{aligned} \tilde{H}_2: \quad \dot{x}_1 &= x_2 & v^T((\alpha-b)v + 0) &> 0 \\ \dot{x}_2 &= -h(x_1) + b\dot{x}_2 - b\dot{x}_2 - \bar{b}x_2 + \tilde{e}_1 & \checkmark \\ \tilde{y}_1 &= x_2 \end{aligned}$$

Also, for $\tilde{e}_1 = 0$, $\tilde{y}_1 = 0 \Leftrightarrow x_2 = 0 \Leftrightarrow h(x_1) = 0 \Leftrightarrow x_1 = 0$

and

$\Rightarrow \tilde{H}_2$ is zero state observable.

$$\tilde{H}_2 \in [0, \beta -$$

\tilde{H}_1 : output strictly passive & zero state observable

$$\rightarrow \tilde{H}_2 \in [0, \infty]$$

\tilde{H}_2 : passive memoryless function

$$\rightarrow \tilde{e}_2 \tilde{y}_2 \geq 0$$

\Rightarrow Thm 6.4 The origin is (globally) asy. stable. (globally since V_1 is radially unbdd)